

Exam Mathematical Models for the spread of infectious diseases (WMMA061-05) Monday January 26 2026, 11.45-13.45

This exam consists of 3 exercises all with subquestions. You can get 90 points in total and the grade for this written exam is obtained through

$$1 + \frac{\text{obtained points}}{10}.$$

Your final grade for the course depends on this grade, the grade for the paper-presentation and (if beneficial) the grade of the homework assignment.

Write your name and student number on every page you hand in and number the pages. Justify your answers, in some cases partial answers may be worth points.

1. Consider a variant of a Markov SIR epidemic in a homogeneously mixing population of $n + 1$ people (with $n \in \mathbb{N}$). The infectious period consists of two stages. During the first stage, the per pair infectious contact rate is λ_1/n , while the length of this stage is exponentially distributed with expectation $1/\gamma_1$. During the second, the per pair infectious contact rate is λ_2/n , while the length of this stage is exponentially distributed with expectation $1/\gamma_2$. If a susceptible and infectious person meet, the susceptible one immediately becomes infectious. After this infectious period, a person is immune/recovered and stays so forever. The processes at which infectious contacts are made and lengths of stages of infectious periods are all independent.

Assume that initially there are n susceptibles and 1 infectious person and there are no recovered people.

a) Show that the number of infectious contacts made by an infected person during their infectious period is distributed as $X_1 + X_2$, where

$$\mathbb{P}(X_j = k) = p_j(1 - p_j)^k \quad \text{for } k \in \{0, 1, 2, \dots\} \text{ and } j \in \{1, 2\},$$

where $p_j = \gamma_j/(\gamma_j + \lambda_j)$ for $j \in \{1, 2\}$ and X_1 and X_2 are independent. Provide a condition (in terms of $\lambda_1, \lambda_2, \gamma_1$ and γ_2) for the epidemic to be supercritical. (15pt)

b) Show that $1 - q$, the asymptotic (as $n \rightarrow \infty$) probability of a large outbreak, is given through the smallest $q > 0$ that satisfies

$$q[1 - q + qp_1][1 - q + qp_2] = p_1p_2.$$

(10pt)

2. Consider a Markov SIR epidemic in a homogeneously mixing population of $n + m$ people (with $n, m \in \mathbb{N}$). The per pair infectious contact rate is $\lambda/(n + m - 1)$. If a susceptible and infectious person meet, the susceptible one immediately becomes infectious. An infectious person stays infectious for an exponentially distributed random period, with expectation $1/\gamma$ (i.e. the per person “rate” of recovery is γ). After this infectious period, a person is immune/recovered and stays so forever. The processes at which infectious contacts are made and infectious periods are all independent.

Assume that initially there are n susceptibles and m infectious people, while there are no recovered people.

Note that if $m = 1$, then this model is effectively the special case of the model of Problem 1, with $\lambda_1 = \lambda$, $\gamma_1 = \gamma$ and $\lambda_2 = 0$.

a) For $m = 1$, compute the Malthusian parameter (asymptotic real time growth rate) α of a supercritical epidemic that leads to a large outbreak. (10pt)

b) Now set $m = cn$, where $c \in (0, 1)$ is a constant. Provide an identity (in terms of λ , γ and c) from which the asymptotic (as $n \rightarrow \infty$) fraction of the initial susceptibles that stays susceptible can be deduced and give a (possibly heuristic) justification of your answer. Here you may assume without proof that the this asymptotic fraction is not random. (15pt)

3. Consider an SIR epidemic in a structured population of n people ($n \in \mathbb{N}$), where “friendships” in the population are modeled according to a configuration model in which asymptotically (as $n \rightarrow \infty$) half of the people have 2 friends and half of the people have 3 friends. Assume that contacts can only occur between friends and pairs of friends contact each other according to independent homogeneous Poisson processes with rate λ . If an infectious person contacts a susceptible one, the susceptible one immediately becomes infectious. An infectious person stays infectious for exactly one time unit. After this infectious period, a person is immune/recovered and stays so forever. Assume that initially there is one infected person of degree 2, and all other people in the population are susceptible.

a) Compute p , the probability that the initial infectious person contacts a given neighbour before he or she recovers. (5pt)

In what follows you may use p as a parameter of the model and express your answers in terms of p .

b) Provide a threshold parameter (e.g. the basic reproduction number) for this epidemic, such that a major outbreak has strictly positive probability if and only if this threshold parameter is strictly larger than 1. (10pt)

c) Recall that the initial infectious individual has degree 2. Compute the asymptotic (as $n \rightarrow \infty$) probability of a large outbreak. (15pt)

Assume that the epidemic is supercritical. Further assume that there is a vaccine available which provides perfect immunity against the disease.

d) If the people that get the vaccine all have degree 3, what is the minimal fraction of the people with degree 3 that should be vaccinated to reduce the probability of a large outbreak to 0?

HINT: The probability of a large outbreak does not change if the vaccine prevents vaccinated individuals from infecting others, instead of preventing them from getting infected themselves. (10pt)

Good luck!